9.4 Application of Fourier Series

$$m x'' + c x' + k x = F(t)$$

↑ mass ↑ damping constant ↖ spring constant

Solution: $x(t) = x_c(t) + x_p(t)$ ← particular (due to input $F(t)$)

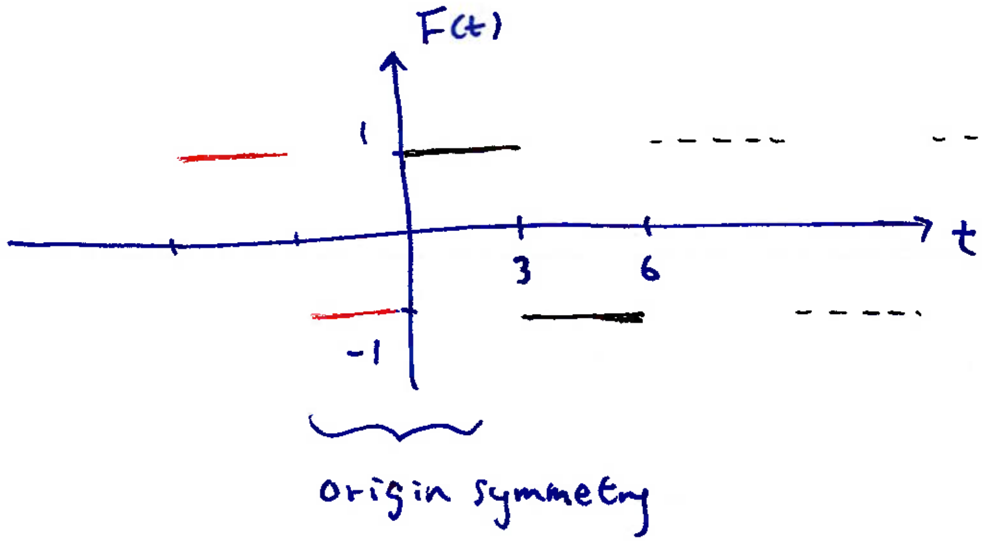
↑ complementary (w/ right side 0)

if $F(t) = F_0 \sin(\omega t)$ undetermined coeff.

$$x_p = A\cos(\omega t) + B\sin(\omega t)$$

if $\omega \neq \sqrt{\frac{k}{m}}$

if $F(t)$ is periodic, ~~decu~~ break into cosines & sines example: $c=0,\ m=1,\ k=5$ $F(t) = \begin{cases} 1 & 0<t<3 \\ -1 & 3<t<6 \end{cases}$ period 6



F(t) is a sine series $a_n = 0,\quad b_n = \frac{2}{L}\int_0^L F(t)\sin\left(\frac{n\pi t}{L}\right)dt$

⋮

$$F(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi t}{3}\right)$$

$$= \frac{4}{\pi}\sin\left(\frac{\pi t}{3}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi t}{3}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi t}{3}\right) + \cdots$$

$x'' + 5x = F(t)$ ←

each term in F(t) results in particular solution

$$x_p = A_n \cos\left(\frac{n\pi t}{3}\right) + B_n \sin\left(\frac{n\pi t}{3}\right) \qquad n = 1, 2, 3, \dots$$

the whole particular solution is $x_p = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{3}\right) + B_n \sin\left(\frac{n\pi t}{3}\right)$

that x_p must satisfy $x'' + 5x = F(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right] \sin\left(\frac{n\pi t}{3}\right)$

sub x_p into ⟶

⋮

$A_n = 0$ for all n

$$B_n = \frac{18\left[1-(-1)^n\right]}{n\pi(45 - n^2\pi^2)}$$

general solution:

$$x(t) = \underbrace{c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t)}_{x_c} + \underbrace{\sum_{n=1}^{\infty} \frac{18\left[1-(-1)^n\right]}{n\pi(45-n^2\pi^2)} \sin\left(\frac{n\pi t}{3}\right)}_{x_p}$$

also steady-periodic solution

$45 - n^2\pi^2 \neq 0$ for $n = 1, 2, 3, \ldots$

close to 0 w/ $n = 2$ because it nearly matches the natural frequency $\sqrt{5} \approx 2.236$

input freq $\frac{n\pi}{3} = \frac{2\pi}{3} \approx 2.094$

(near resonance)

for this one, it's ok because if $n = 2$, $B_n = 0$ because $[1 - (-1)^n]$

if we had $x'' + 5x = \begin{cases} 1 & 0 < t < 10 \\ -1 & 10 < t < 20 \end{cases}$ period 20

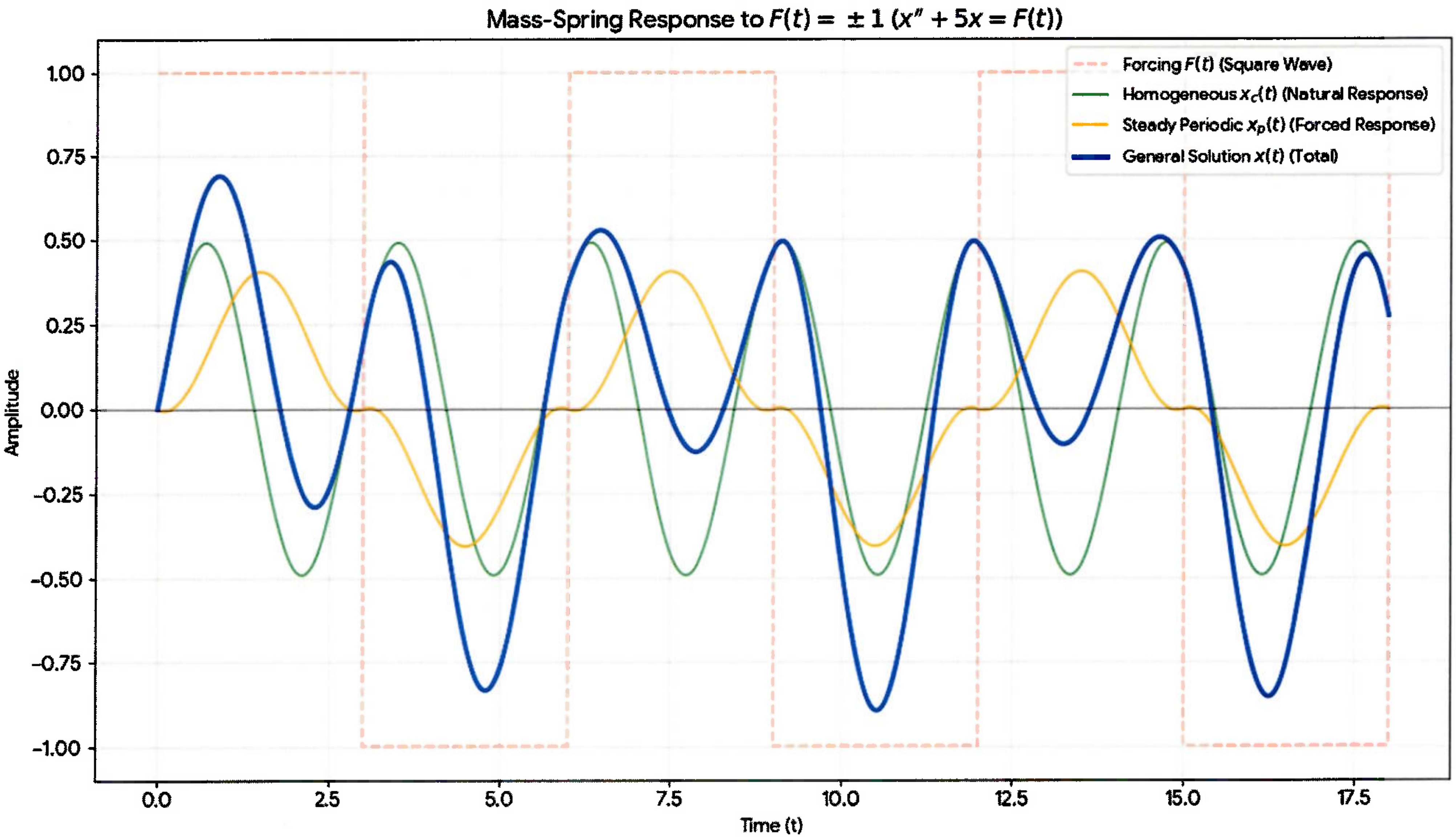
⋮

$$x_p = \sum_{n=1}^{\infty} \frac{200\,[1-(-1)^n]}{n\pi(500 - n^2\pi^2)} \sin\left(\frac{n\pi t}{10}\right)$$

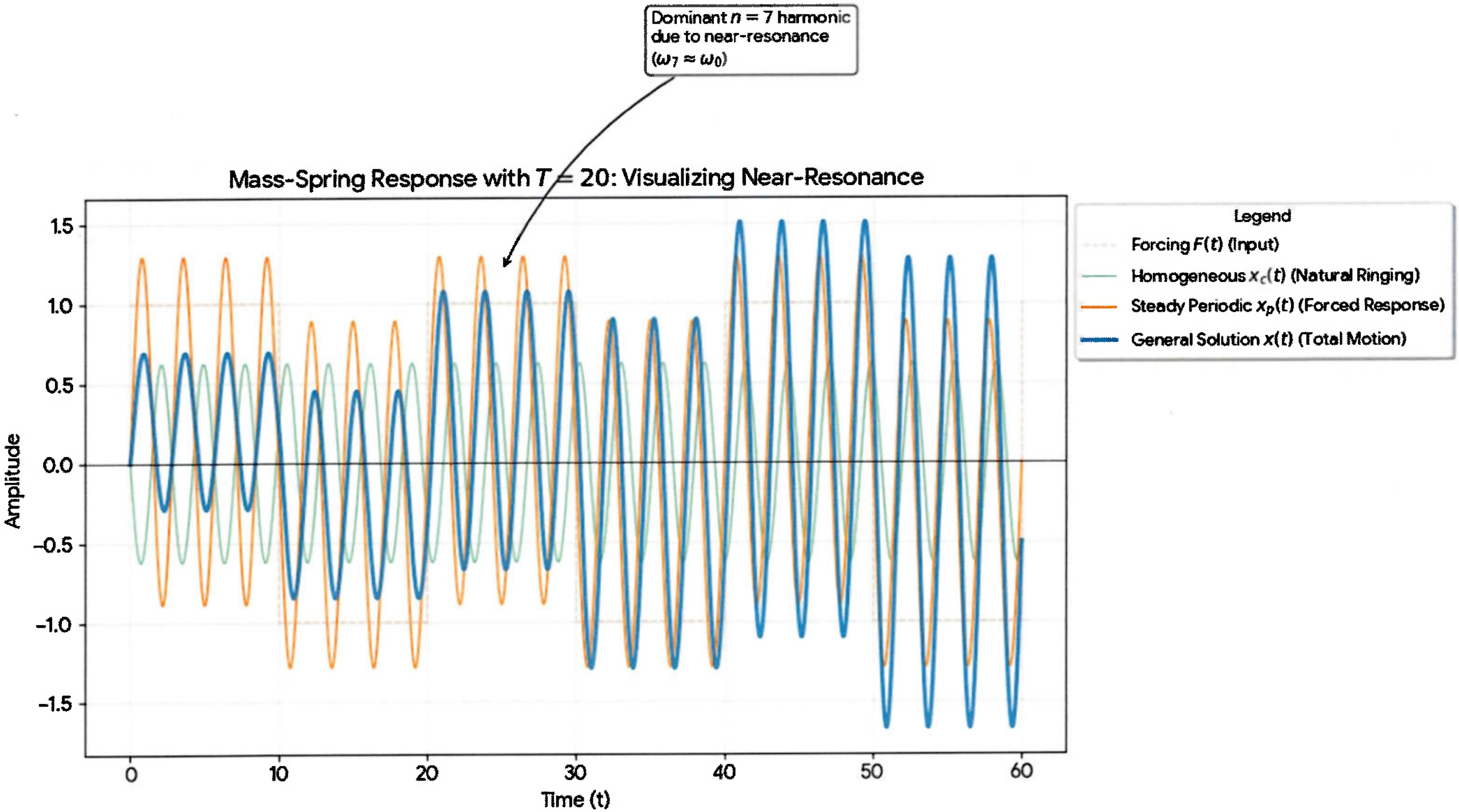
$500 - n^2\pi^2$ is close to 0 w/ $n = 7$

$[1 - (-1)^n]$ does NOT eliminate it

near resonance is visible



Mass-Spring Response to F(t) = ±1 (x″ + 5x = F(t))
Forcing F(t) (Square Wave)
Homogeneous x_c(t) (Natural Response)
Steady Periodic x_p(t) (Forced Response)
General Solution x(t) (Total)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Amplitude
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
Time (t)


Dominant $n = 7$ harmonic
due to near-resonance
$(\omega_7 \approx \omega_0)$
Mass-Spring Response with $T = 20$: Visualizing Near-Resonance
Legend
Forcing $F(t)$ (Input)
Homogeneous $x_c(t)$ (Natural Ringing)
Steady Periodic $x_p(t)$ (Forced Response)
General Solution $x(t)$ (Total Motion)
Amplitude
Time (t)
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
10
20
30
40
50
60

now we look at $x'' + 9x = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$ period 2π

$\vdots$

$$x_p = \sum_{n=1}^{\infty} \frac{2\left[1-(-1)^n\right]}{n(9-n^2)} \sin(nt)$$

$n = 3$ makes $9 - n^2 = 0$ pure resonance

the series solution x_p is not valid for $n = 3$

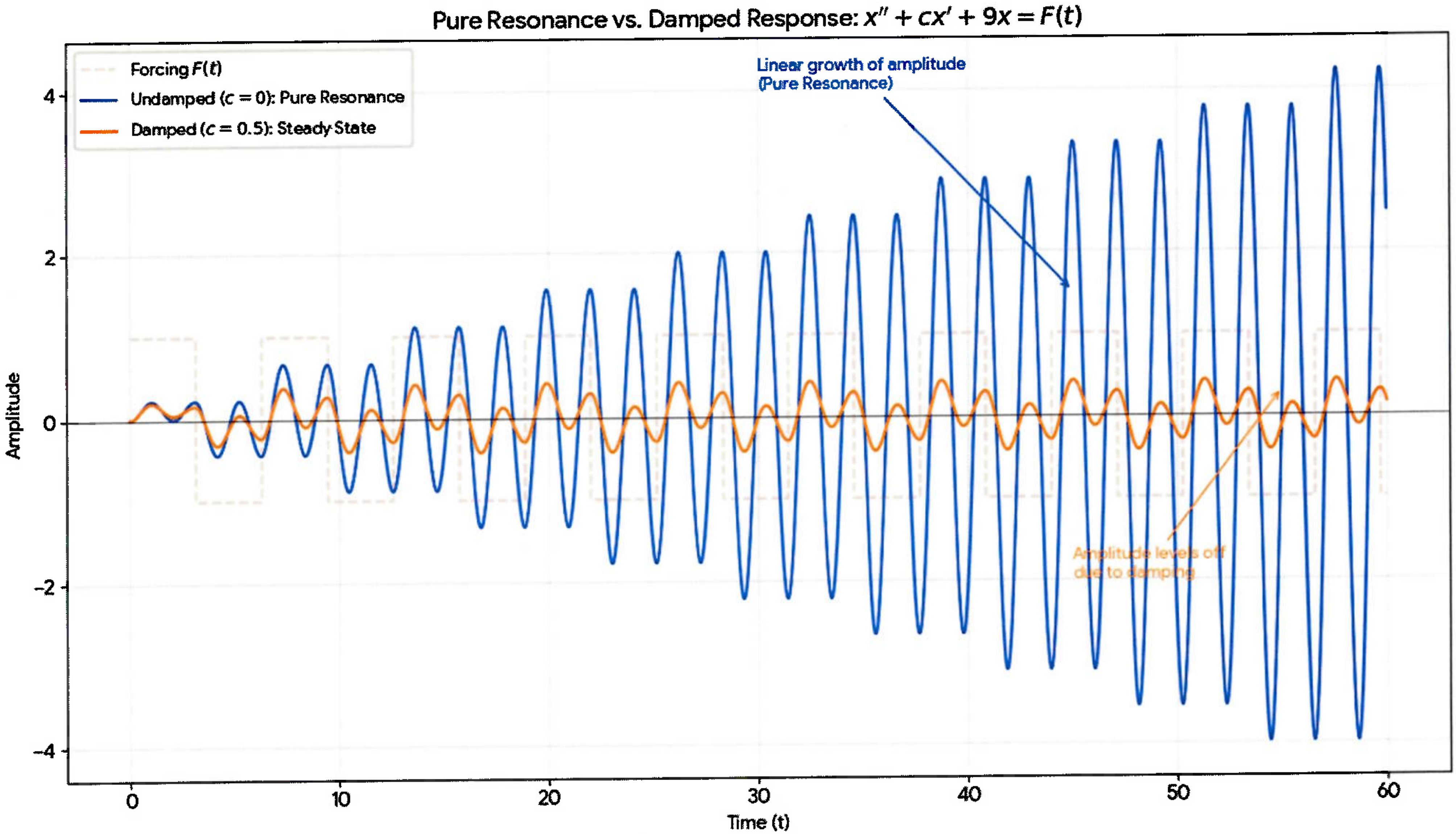
consider that one term separately

$$\underbrace{x'' + 9x}_{\substack{\cos(3t)\\ \sin(3t)}} = \underbrace{At\cos(3t) + Bt\sin(3t)}_{x_p}$$

(complementary)

$\vdots$

$A = -\frac{2}{3}$ $\quad B = 0$


Pure Resonance vs. Damped Response: $x'' + cx' + 9x = F(t)$
Forcing $F(t)$
Undamped ($c = 0$): Pure Resonance
Damped ($c = 0.5$): Steady State
Linear growth of amplitude
(Pure Resonance)
Amplitude levels off
due to damping
4
2
0
−2
−4
Amplitude
0
10
20
30
40
50
60
Time (t)

General solution:

$$x(t) = C_1\cos(3t) + C_2\sin(3t) - \frac{2}{3}t\cos(3t) + \sum_{\substack{n=1\\ n\neq 3}}^{\infty} \frac{2[1-(-1)^n]}{n(9-n^2)}\sin(nt)$$

if $c \neq 0$, $mx'' + cx' + kx = F(t)$

$$F(t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi t}{L}\right)$$

$\vdots$

$$x_p = \sum_{n=1}^{\infty} \frac{F_n}{\sqrt{(k-m\omega_n^2)^2 + (c\omega_n)^2}} \sin\left(\frac{n\pi t}{L} - \tan^{-1}\left(\frac{c\omega_n}{k-m\omega_n^2}\right)\right)$$

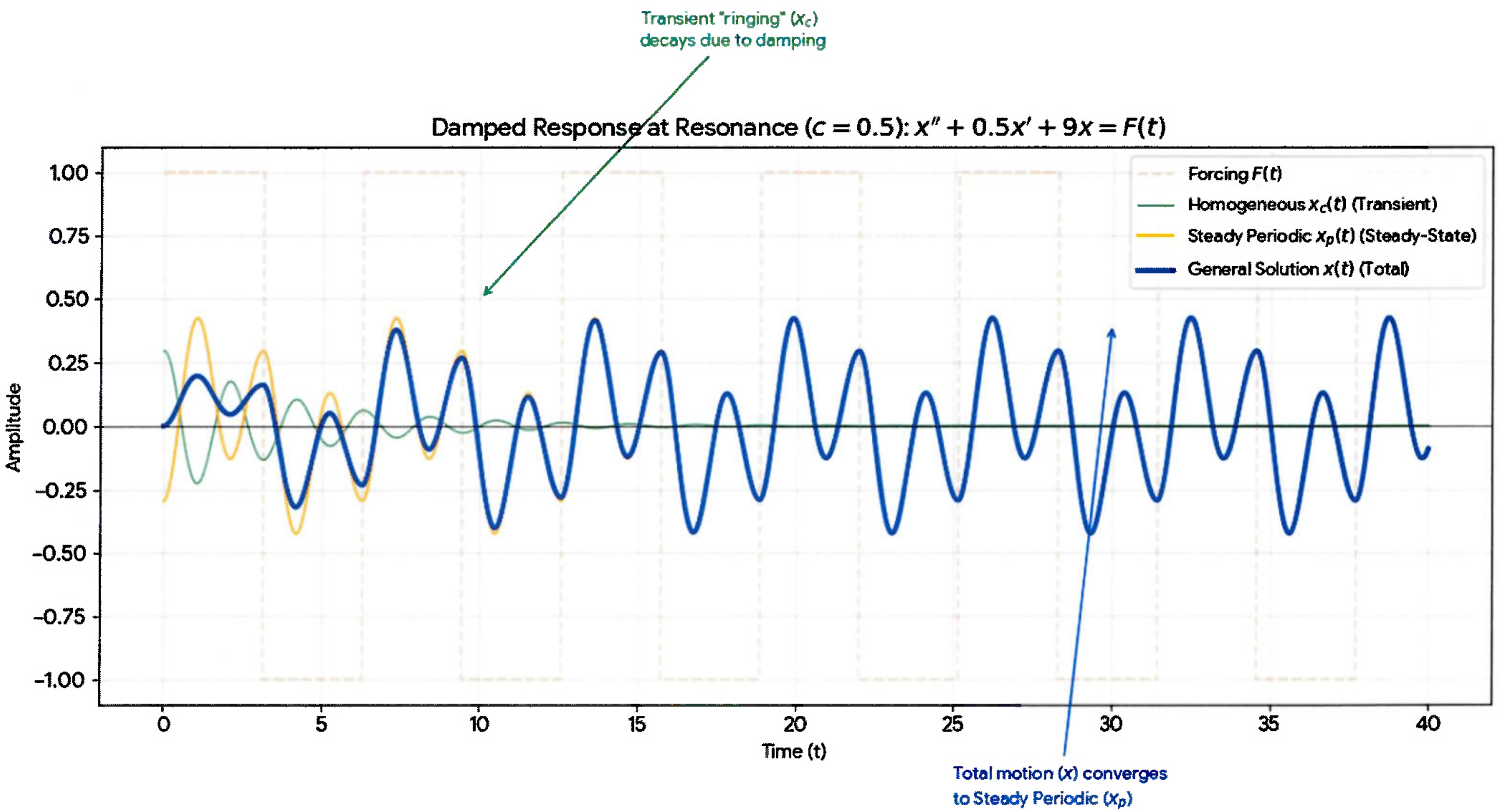
$$\omega_n = \cancel{\sqrt{\frac{k}{m}}}\,\frac{n\pi}{L}$$

if $c \neq 0$

denom $\neq 0$

damper introduces a phase shift

(steady periodic lags complementary)



Transient "ringing" (x_c)
decays due to damping
Damped Response at Resonance ($c = 0.5$): $x'' + 0.5x' + 9x = F(t)$
Forcing $F(t)$
Homogeneous $x_c(t)$ (Transient)
Steady Periodic $x_p(t)$ (Steady-State)
General Solution $x(t)$ (Total)
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
Amplitude
0
5
10
15
20
25
30
35
40
Time (t)
Total motion (x) converges
to Steady Periodic (x_p)